

When Does Successful Comparison of Decimals Reflect Conceptual Understanding?

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Forty-eight students across Grades 3 to 6 were interviewed individually using a range of tasks, where the mathematical focus was decimal knowledge and understanding. Students who may be categorised as “apparent experts” on a decimal comparison test were found to differ considerably in their ability to perform ordering and benchmarking tasks. Those students whose explanations when comparing decimals reflected a greater place value knowledge and who were not following a rule which ultimately treats decimals as whole numbers, appeared to have a more conceptual understanding of the decimal numeration system and were able to apply this understanding to more difficult (or novel) tasks.

A connected understanding of decimals has increased in importance since the introduction of the metric system of measurement and the wider use of calculators and computers. Results from major studies (e.g., Brown, 1981; Wearne & Kouba, 2000) indicated that decimals create great confusion for many students and many studies indicated that much of this difficulty arose because students were treating decimals as whole numbers. It has also been documented that students rely on procedures to the detriment of number sense or meaning when computing with decimals (Hiebert & Wearne, 1985).

Over the last two decades there has been considerable documentation of erroneous rules or misconceptions that students appear to use when asked to compare or order decimal numbers. Naming, defining and fine tuning these rules or codes have been the focus of much research (Archer & Condon, 1999; Irwin, 1996; Moloney & Stacey, 1996; Resnick, Neshet, Leonard, Magone, Omanson & Peled, 1989; Sackur-Grisvard & Leonard, 1985; Stacey, Helme & Steinle, 2001; Stacey & Steinle, 1998, 1999; Steinle & Stacey, 2001, 2002). The terms have changed, the definitions varied and where there were once three (Sackur-Grisvard & Leonard, 1985) there are now twelve (Stacey & Steinle, 1998).

A considerable body of research exists on students’ understanding of decimals and the prevalence and persistence of misconceptions (Steinle & Stacey, 2003). Much of this knowledge has been inferred from responses to pencil and paper decimal comparison tests. When students are asked to choose the larger of two decimals they are commonly categorised as using one of three erroneous rules. Sackur-Grisvard and Leonard (1985) defined three systematic but incorrect rules that fourth and fifth grade French students used to decide which decimal number was greater:

- Rule One: The number with the more decimal places is the larger; e.g., 3.214 is greater than 3.8 because $214 > 8$. This rule was fairly common; around 40% of fourth graders and 10% of seventh graders used this rule.
- Rule Two: The number with the fewer decimal places is the larger; e.g., 1.2 is larger than 1.35 because they believe tenths are always larger than hundredths. This rule was the least common; less than 6% using the rule in all grades.
- Rule Three: A correct judgement is given if there is a zero immediately to the right of the decimal point in one of the decimals being compared, but otherwise choose as for rule one. This rule remained reasonably constant, being used by between 7% and 13% across the four grades.

Resnick, Nesher, Leonard, Magone, Omanson and Peled (1989) renamed these rules, calling Rule One the “whole number rule”, Rule Two the “fraction rule”, and Rule Three the “zero rule”, and developed a decimal comparison test of eight pairs of decimal numbers (including two fractions). Moloney and Stacey (1996) developed a pencil and paper test of 15 pairs of decimals items. These items were largely taken from Resnick et al. (1989), except that the fraction tasks were replaced with decimals.

Stacey and Steinle (1998) developed a new test extending that of Moloney and Stacey (1996) that had 14 core items and 11 supplementary items. This test took the categories of “Longer is Larger” (previously whole number rule), “Shorter is Larger” (previously fraction rule), “Zero Rule” and “Expert Rule” and classified these in a more refined way. By 1999, Stacey and Steinle renamed the “expert” category as the “apparent experts” or “task expert”, claiming these students “may possess excellent understanding or may apply correct rules not understood or may have one identified incorrect pattern of thinking” (p. 446).

The focus of this paper is to consider the implications for students who are categorised as “apparent experts” on a decimal comparison test but who achieve this status by using a rule by which zeros are added to equalise the length of the shorter decimal and then compare as whole numbers. Resnick et al. (1989) suggested that students who are taught to add zeros have remained at the conceptual level attributed to whole-number-rule students and that “such syntactic teaching would serve to suppress errors in performance without improving children’s conceptual understanding” (p. 26).

Little has been written about the reasoning behind students who have been classified as apparent experts and the consequences of this reasoning in working on more difficult (or novel) tasks. This aspect formed the basis of part of the present study, and is the focus of this paper.

The Present Study

The purpose of the study was to investigate students’ understandings (and misunderstandings) about decimals, using a one-to-one, task-based interview. This paper focuses on those particular tasks which uncover a range of understandings or strategies of students who perform accurately on a decimal comparison test, and the consequences of this understanding on more difficult (or novel) tasks.

The participants were 48 children from a middle-class, co-educational Catholic primary school in suburban Melbourne. The students included 8 from Grade 3, 12 from Grade 4, 19 from Grade 5 and 9 from Grade 6. Information letters and consent forms were sent home to around 300 students, and then a sample of those students whose parents responded positively was interviewed. General achievement in mathematics was not a criterion, although there was clearly a considerable range of levels of mathematical understanding across the 48 children. Each child was interviewed for around 30 minutes on an assortment of tasks. No child was asked all questions in the set. The three tasks that will be outlined are taken from a much larger set of questions within the original interview.

The Original Interview

The original interview was created by taking or adapting tasks collected from the review of the literature. These tasks were sorted into some of the key ideas (and related skills) that emerged from the literature.

Some of the key ideas (and related skills) about decimals that were explored during the original interview were:

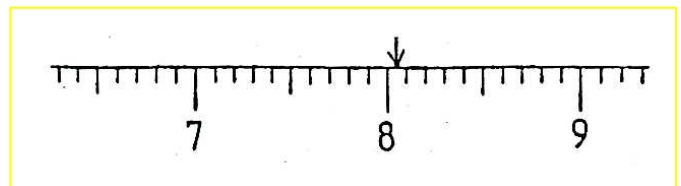
- Place Value (including reading, writing, interpreting the place value of a decimal and counting patterns)
- Representations of decimals (interpreting different models)
- The size of decimals (comparing and ordering decimals and using benchmarks)
- The density of decimals (identifying decimals between two decimals)
- Equivalency and relationships (identifying equivalency in decimals and relationships between decimals, percents and fractions)
- Applications (operations and estimating)

A small sample of the tasks from the original interview is outlined below.

Place value. A set of cards that includes 0, 1, 2, 7, a decimal point, and two blanks, is placed in front of the student who is asked to use the cards to show the interviewer what “two tenths” would look like. The interview continues with the following numbers.

- | | | |
|----------------------|---------------------|-----------------------|
| (a) two tenths | (d) 2 thousandths | (g) 27 tenths |
| (b) seven hundredths | (e) 702 thousandths | (h) one and 17 tenths |
| (c) 27 hundredths | (f) ten tenths | (i) 712 hundredths |

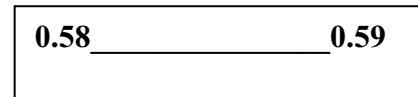
Representations of decimals. The student is asked to indicate which number the arrow is indicating on the number line.



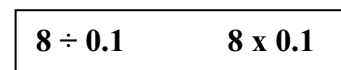
The student is asked to name the decimal that is represented by the shaded region.



The density of decimals. The student is asked if any decimals lie between 0.58 and 0.59. If “yes” they are asked to name one and to indicate how many decimals might lie between these two numbers.



Applications. The student is asked which number would give the bigger answer and to explain why.



Tasks Assessing Relative Size of Decimals

A set of decimal pairs (see Fig. 1) was selected to attempt to unearth common misconceptions already identified in the literature.

Task 1.

In the first task the student was asked to compare two decimal numbers and say which was larger and why.

1.973	19.73	0.567	0.3
0.4	0.3	4.32	4.3278
1.45	1.46	0.087	0.87
0.37	0.217	0	0.6
		0.7	0.70

Figure 1. A multi-stage task comparing the size of two decimal numbers.

In the next task (see Fig. 2) the students were required to order a set of twelve decimals. Steinle and Stacey (2001) stated that “it has been established that comparisons of pairs is simpler for students, reducing the information processing demands, yet can reveal as much” (p. 434). This may be referring to pairs as compared to triples of decimals as first used by Sackur-Grisvard and Leonard (1985). It might however be that the more demanding task of a much larger set may unearth latent erroneous thinking.

Task 2.

Twelve number cards are arranged randomly in front of the student who is asked to order them from smallest to largest.

0	0.01	0.10	.356	0.9	1	1.2	1.7	2	1.70	1.05	.10
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Figure 2. A task involving ordering a set of twelve decimal numbers.

The third task (see Fig. 3) was designed to elicit a response that most likely requires benchmarking. “Meaningful decimal fraction ideas are like good mental maps. For example, if you have an accurate mental map of a neighborhood you can “feel” your way to an unknown place within it. It’s “close to” or “far from” certain landmarks” (Langford & Sarullo, 1993, p. 241). Others (e.g., McIntosh, Bana & Farrell, 1997; McIntosh, Reys & Reys, 1992; Markovits & Sowder, 1994) have referred to this concept as benchmarking. McIntosh, Reys and Reys (1997) believe that “the variety and complexity of the benchmarks in making decisions about numbers and numerical contexts, is a valuable indicator of number sense” (p. 6).

Markovits and Sowder (1994) conducted an intervention program with seventh grade students for the purpose of developing number sense. Results from a test following the program showed that students were less likely to use memorised procedures, such as adding enough zeros to compare uneven decimals, or converting fractions to common denominators, and were more likely to use benchmarks when comparing decimals with fractions.

Task 3

In this task the student is presented with two cards and is asked which of these numbers (pointing to the string of numbers) is closest to this (pointing to 0.18).

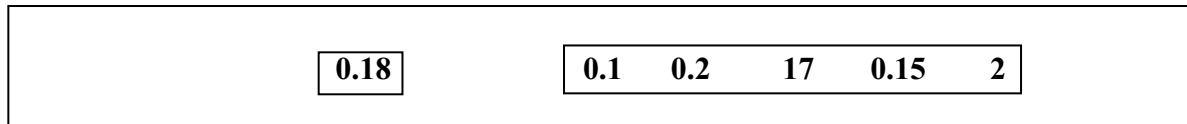


Figure 3. A task involving benchmarking the size of decimals.

Results and Discussion

In relation to the responses to the tasks in the present study, we have suggested that students who have no more than one error in the comparison test (Task 1), are possibly “apparent experts” or “task experts” as defined by Stacey and Steinle (1998, 1999). As the set of decimal pairs was not the same as those used by Stacey and Steinle, this assumption may be incorrect for some students. The status of apparent expert has been identified by *patterns of behaviours* on a pencil and paper test; that is by the number of correct answers. *Ways of thinking* may be inferred from the explanations given in response to their particular selection or behaviour.

Different ways of thinking were categorised for those students who achieved no more than one error on the decimal comparison set and who were assigned the status of “apparent expert”. This thinking generally fell into two groups:

1. Justification that we have termed Place Value Judgement (PVJ) are those who used fractional language (apart from those explanations that could be categorised as shorter is larger thinking) and benchmarking.
2. Justification that we have termed Whole Number Judgement (WNJ) predominantly used whole number language including those who used the rule for extending decimals of uneven length by adding zeros.

Examples of Place Value Judgement are:

- 1.46 is greater than 1.45 because 1.46 is one hundredth more
- 0.567 is greater than 0.3 because five tenths is greater than three tenths, or 0.567 is more than one half but 0.3 is less than a half
- 0.87 is greater than 0.087 because 87 hundredths is greater than 87 thousandths
- 0.7 is the same as 0.70 because 7 tenths equals 7 tenths (or 70 hundredths)

Examples of Whole Number Judgement are:

- $0.4 > 0.3$ because four is greater than three
- $0.567 > 0.3$ because $567 > 300$

In Task 1, each pair was selected to uncover a pattern of behaviour or pattern of thinking identified in the literature. Asking the students to justify their choice provided a window into their thinking for each pair. Some students were found to use a particular pattern or rule consistently, while others changed from pair to pair.

The most common misconceptions uncovered through this interview were:

- WNT: Whole Number Thinking; e.g., $0.217 > 0.37$ because two hundred and seventeen is greater than thirty-seven
- LILT: Longer is Larger Thinking; e.g., $0.217 > 0.37$ because this (0.217) has more numbers

- SILT: Shorter is Larger Thinking; e.g., $0.3 > 0.567$ because tenths are larger than thousandths
- RT: Reciprocal Thinking; e.g., $0.3 > 0.4$ because $\frac{1}{3} > \frac{1}{4}$
- DLZ: Decimals are less than zero; e.g., $0 > 0.6$

Two patterns of thinking that consistently provided correct responses were Place Value Judgement (PVJ) and Whole Number Judgement (WNJ).

Students who made no more than one error in the set of nine pairs of decimals were grouped as “apparent experts” and then subdivided into those that used predominantly PVJ or WNJ.

Twenty-four students were asked to order the set of twelve decimals in Task 2 and four of these ordered the set correctly. Only two fourth graders (Paula and David) were asked this question. They both would be classified as “apparent experts” due to their number of correct responses in Task 1. Paula would add zeros and then compare decimals as whole numbers (and had no errors in Task 1), while David would use a Place Value Judgement (with some benchmarking, e.g., 0.37 is closer to four tenths than 0.217) (with one error in Task 1). Their results for Task 2 were:

Paula:
 0 0.01 1.05 0.9 .10 0.10 .356 1 1.2 1.7 1.70 2

David:
 0 0.01 .10 .356 0.9 1 1.05 1.2 1.70
 0 0.01 0.10 .356 0.9 1 1.05 1.2 1.7 2

Figure 4. Results from Task 2: Ordering a set of twelve decimal numbers.

The boxed sets indicate David placed one above the other and said that they were “the same”. David appears to be able use his conceptual understanding of decimals to correctly complete this task. Paula on the other hand did not appear to notice the equivalent decimals, and appears to be ordering the numbers using a whole number rule such that “point nine comes before point ten”. Maybe the integer in 1.05 was ignored and the ordering from zero, in her mind, was 1, 5, 9, 10, 10, 356.

These two examples suggest that Paula’s ability to compare pairs of decimals with perfect accuracy was not an indication of a stable understanding of the relative size of decimals. On the other hand, David’s use of place value and fraction knowledge meant that he was not distracted by the number of digits or misinterpreted them as whole numbers.

For Task 3 only four out of 17 students correctly answered “0.2 is two hundredths away from 0.18”. All of these were apparent experts from Task 1 and were further categorised as predominantly using PVJ in their explanations. Common errors were “0.15 because point fifteen is closest to point eighteen” and “17, because seventeen is closest to eighteen”, indicating that students were predominantly treating decimals as whole numbers.

Summary and Implications

The present study was designed to assess student understanding of decimals through a task-based interview. In this paper the performance of students on three particular tasks was discussed. These tasks follow the progress of students who may be assigned as “apparent experts” on a decimal comparison task. It was found that students who obtained the status of apparent expert by using a rule where zeros are added to equalise the length of

two decimals and then compared as whole numbers; were not able to demonstrate a stable understanding of the relative size of decimals on more difficult (or novel) tasks.

Implications for both teaching and research that emerge from this study include the following:

- teaching students to annex zeros before comparing decimals may be to the detriment of their conceptual understanding;
- ordering more than two decimals is likely to prove a more searching task and a greater indicator of a stable understanding of the relative size of decimals than merely looking at a comparison of pairs;
- using fractional language to describe decimals more often may contribute to a clearer conception of the decimal numeration system (i.e., encouraging students to describe 2.75 as “2 and 75 hundredths” rather than “2 point 75”);
- assessing ways of thinking may give a clearer picture of a students’ understanding than assessing patterns of behaviour on a pencil and paper test.

These results add further weight to the calls by many researchers (e.g., Skemp, 1976; Wearne & Hiebert, 1986) for teaching which focuses on conceptual understanding to a far greater extent than procedural understanding.

The challenge for researchers remains to communicate these findings in a form that is accessible to a teaching population, many of whom lack confidence in their own knowledge of this important area.

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